Vector Geometry

In this document we consider some of the geometrical properties of vectors, that relates to the use of vectors in applied mathematics[1] and physics[2].

Distance between two vectors

The geometrical distance between two vectors **x** and **y** is written $|\boldsymbol{y}-\boldsymbol{x}|$can be found by Pythagoras theorem.

In two dimensions: $|\boldsymbol{y}-\boldsymbol{x}| = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$

In three dimensions: $|\boldsymbol{y}-\boldsymbol{x}| = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}$

Example 1

Find the distance between the vectors $\binom{2}{4}$ and $\binom{5}{8}$.

Answer

$$\left|\binom{5}{8}-\binom{2}{4}\right| = \sqrt{(5-2)^2+(8-4)^2} = 5$$

Example 2

Find the distance between the vectors $\begin{pmatrix}2\\3\\5\end{pmatrix}and\begin{pmatrix}6\\7\\7\end{pmatrix}$.

Answer

$$\left|\begin{pmatrix}6\\7\\7\end{pmatrix}-\begin{pmatrix}2\\3\\5\end{pmatrix}\right| = \sqrt{(6-2)^2+(7-3)^2+(7-5)^2} = 6$$

[1] Vector Arithmetic
[2] Vectors and Scalars

The size of a vector

The size of a vector is equivalent to the geometrical distance between the origin and the vector:

$$|\boldsymbol{x} - 0| = |\boldsymbol{x}|.$$

In two dimensions: $|\boldsymbol{x}| = \sqrt{(x_1)^2 + (x_2)^2}$

In three dimensions: $|\boldsymbol{x}| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$

The size of a vector is equivalent to its 2-norm[3]

Example 3

Find the size of the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Answer

$$\left|\begin{matrix} 3 \\ 4 \end{matrix}\right| = \sqrt{3^2 + 4^2} = 5$$

Example 4

Find the size of the vector $\begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$.

Answer

$$\left|\begin{matrix} 4 \\ 4 \\ 2 \end{matrix}\right| = \sqrt{4^2 + 4^2 + 2^2} = 6$$

[3] Vector Norm and Normalisation

Area of a triangle joining three points in 3D

Let **x**, **y** and **z** be three points in three-dimensional space. The area of the triangle that links the three points is given by the formula

$$\text{Area} = \frac{1}{2}\sqrt{\{(\boldsymbol{y}-\boldsymbol{x}).(\boldsymbol{y}-\boldsymbol{x})\}\{(\boldsymbol{z}-\boldsymbol{x}).(\boldsymbol{z}-\boldsymbol{x})\}-(\boldsymbol{y}-\boldsymbol{x}).(\boldsymbol{z}-\boldsymbol{x})}\ .$$

Normal to a line between two vectors in 2D

Consider the line joining two 2-vectors **x** to **y**. The direction of the line is **z** = **y**-**x**. The normal to the line has the direction $\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}$.

The unit normal to the line is given by $\frac{1}{|\boldsymbol{z}|}\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}$.

Note that the normal may also be defined in the opposite direction.

Normal to a plane in 3D space

Let **x**, **y** and **z** be three (non-co-linear) points in three-dimensional space. Then the vector (**y**-**x**)×(**z**-**x**) is a normal to the plane. Note that the vectors **x, y** and **z** can be interchanged and the resulting vector is still a normal.

Dividing the resulting normal by its size gives a unit normal. That is

$$\frac{(\mathbf{y}-\mathbf{x})\times(\mathbf{z}-\mathbf{x})}{|(\mathbf{y}-\mathbf{x})\times(\mathbf{z}-\mathbf{x})|}$$

is a unit normal to the plane.

Note that the normal may also be defined in the opposite direction.