Surfaces

Consider the set of points $\boldsymbol{r} = (x(u,v), y(u,v), z(u,v))$ where u and v are continuous parameters, taking all values in the region R in the uv-plane and x, y and z are continuous functions of u,v in R. For every value of (u,v) in R there is a corresponding point (x,y,z) on a surface S in 3D space. Hence we have a mapping from R to the surface S.

Example 1

The surface defined by

$$\boldsymbol{r} = (2\sin(\theta)\cos(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\theta))$$

where $0 \leq \theta \leq \pi$ and $-\pi \leq \varphi \leq \pi$ is the surface of a sphere of radius 2 centred at the origin.

Open and Closed Surfaces

A surface S is said to be *open* if every pair of points that are not on S can be joined by a continuous line which does not cross S.

A surface S is said to be *closed* if it divides space into two distinct regions; an interior region D and an exterior region E. That is that any continuous line joining any point in D with any point in E must cross the surface S at least once.

Example 1

The surface of a sphere defined in example 1 above is clearly a closed surface.

Example 2

The surface defined by

$$\boldsymbol{r} = (2\sin(\theta)\cos(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\theta))$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq \pi$ is a hemispherical cap of radius 2 centred at the origin,

and it is an open surface.

Unit normal vector

For the surface defined above, $\boldsymbol{r} = (x(u,v), y(u,v), z(u,v))$, at any point on the surface, we can define the directional derivatives

$$\frac{\partial \boldsymbol{r}}{\partial u} = \boldsymbol{r_u}\ , \frac{\partial \boldsymbol{r}}{\partial v} = \boldsymbol{r_v}\ ,$$

The vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}$$

is the unit normal to the surface at $\boldsymbol{r}$. (Note that × represents the cross product[1].) The normal to a surface may point in either direction; if $\boldsymbol{n}$ is a normal to a surface then so is $-\boldsymbol{n}$. Normally the surface(s) are labelled so that there is no ambiguity. In the case of a closed surface, then the convention is that the normal points outward from the surface; the outer surface is labelled as the positive surface. A surface is said to be *oriented* once the positive side has been determined.

<u>Example 1</u>

The surface of a sphere defined in example 1 is a closed spherical surface.

$$\boldsymbol{r} = (2\sin(\theta)\cos(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\theta))$$

Hence $\boldsymbol{r}_\theta = (2\cos(\theta)\cos(\varphi), 2\cos(\theta)\sin(\varphi), -2\sin(\theta))$
$\boldsymbol{r}_\varphi = (-2\sin(\theta)\sin(\varphi), 2\sin(\theta)\cos(\varphi), 0)$.

Hence $\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2\cos(\theta)\cos(\varphi) & 2\cos(\theta)\sin(\varphi) & -2\sin(\theta) \\ -2\sin(\theta)\sin(\varphi) & 2\sin(\theta)\cos(\varphi) & 0 \end{vmatrix}$.

$= (4\sin^2(\theta)\cos(\varphi), 4\sin^2(\theta)\sin(\varphi), 4\sin(\theta)\cos(\theta))$

Hence $|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi|^2 = 16\sin^4(\theta)\cos^2(\varphi) + 16\sin^4(\theta)\sin^2(\varphi) + 16\sin^2(\square)\cos^2(\square)$

$= 16\sin^4(\theta) + 16\sin^2(\square)\cos^2(\square) = 16\sin^2(\theta)$.

Hence $|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi| = 4\sin(\theta)$ and

$$\boldsymbol{n} = \frac{\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi}{|\boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi|} = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)).$$

Note that the normal points outward from the surface; hence we do not change its orientation.

<u>Example 2</u>

The surface is the upper hemisphere of the surface already defined in example 1. The normal is defined similarly. However, since this is an open surface, we can define either side as the positive surface. We may reverse the normals, if we wish, as long as the orientation of the normals is applied consistently across the surface.

[1] Vector Arithmetic

If the unit normal is continuous at all points on the surface then the surface is said to be *smooth*. For example the surface of a sphere is a smooth surface. A *simple surface, piecewise smooth* or *regular* surface is one which is composed of a set of smooth surfaces. For example the surface of a cube is a simple surface.

Surface Area

Returning to our general definition of a surface S:

$$\boldsymbol{r} = \big(x(u,v), y(u,v), z(u,v)\big),$$

with u and v taking a range of values so that the whole of S is covered. The surface area is given by the formula

$$\text{surface area of } S = \iint |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, du \, dv$$

Example 1

The surface area of the sphere of example 1 is given by the formula

$$\int_0^{\pi} \int_{-\pi}^{\pi} 4\sin(\theta)\, d\varphi\, d\theta = \int_0^{\pi} 8\pi \sin(\theta) d\theta = 16\,\pi\,.$$