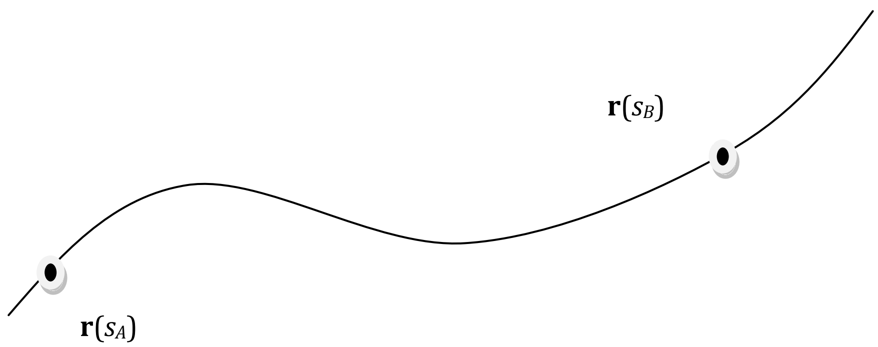
Line Integration in Vector Calculus

In this document we consider the integration[1] of scalar and vector fields along a line. A line will be defined by $\mathbf{r} = \mathbf{r}(s) = \big(x(s), y(s), z(s)\big), (0 \le s \le l)$. Let the range of integration be from the point $\mathbf{r}(s_A)$ to $\mathbf{r}(s_B)$, where $0 \le s_A \le s_B \le l$. The domain of the line integration is illustrated in the following figure.



Line Integral of a Scalar Field

Let $\varphi(\mathbf{x})$ be a scalar field defined at all points on the line. The line integral of φ along the curve **r**, from $\mathbf{r}(s_A)$ to $\mathbf{r}(s_B)$ is defined to be

$$I = \int_{s_A}^{s_B} \varphi\,\big(x(s), y(s), z(s)\big) ds\ .$$

Example

Let **r**= $(3, e^s, (3+s^2)^{1/2})$ and let $\varphi = z^2 - x$, find the line integral of φ along the curve **r**, from $s=1$ to $s=2$.

Answer

$$I = \int_1^2 (z^2 - x)\, ds = \int_1^2 (3+s^2) - 3\ \ ds = \int_1^2 s^2\ \ ds = \left[\frac{s^3}{3}\right]_1^2 = \frac{7}{3}.$$

Integral around a closed curve

The integral around a closed curve can be found in the same way. The result is the same wherever the start/finish point is on the closed curve.

[1] Integration

Example

Integrate the scalar field $\varphi = x^2 + y^2 + z^2$ around the closed curve of a circle of radius 2, centred on the z axis and on the z=3 plane (**r**= $(2\cos(\theta), 2\sin(\theta), 3)$ with ($0 \le \theta \le 2\pi$)). Find the line integral of φ along the curve **r** on the closed curve.

Answer

$$I = \int_0^{2\pi} (x^2 + y^2 + z^2)\, d\theta = \int_0^{2\pi} (4\cos^2(\theta) + 4\sin^2(\theta) + 3^2)\, d\theta = \int_0^{2\pi} 16\ d\theta = 32\pi$$

Line Integral of a Vector Field

Let $\mathbf{F}(x, y, z)$ be a vector field and let $\mathbf{r} = \mathbf{r}(s) = (x(s), y(s), z(s))$, $(0 \le s \le l)$ be a curve C along which we wish to integrate **F**.

Scalar Line Integral

The *scalar line integral* is defined by

$$I = \int_0^l \mathbf{F}.\mathbf{t}\, ds\,,$$

where $\mathbf{t}$=$\mathbf{t}(s)$ is the unit tangent to the line. Since $\mathbf{t} = \frac{d\mathbf{r}}{ds}$, then $\mathbf{t}\, ds = d\mathbf{r}$ and we can re-write the integral above as

$$I = \int_0^l \mathbf{F}.d\mathbf{r}\,.$$

If C is a simple closed curve then we may write

$$I = \oint_C \mathbf{F}.d\mathbf{r}$$

and the integral is referred to as the *circulation* of **F** around C. If the sense of the definition of the curve is reversed then the tangent is also reversed and the sign of the circulation changes.

Example

Find the circulation of the vector field $\boldsymbol{F} = (y, -x, xyz)$ about the ellipse C (**r**= $(\cos(\theta), 2\sin(\theta), 4)$ ($0 \le \theta \le 2\pi$)).

Answer

The circulation is given by the equation $I = \oint_C \mathbf{F}.d\mathbf{r} = \int_0^{2\pi} \mathbf{F}.\frac{d\mathbf{r}}{d\theta}\, d\theta$.

$\frac{dr}{d\theta} = (-\sin(\theta), 2\cos(\theta), 0)$, on C, $\boldsymbol{F} = (2\sin(\theta), -\cos(\theta), 8\sin(\theta)\cos(\theta))$

$$I = \int_0^{2\pi} (-2\sin^2(\theta) - 2\cos^2(\theta))\ d\theta = \int_0^{2\pi} -2\ d\theta = -4\pi\,.$$