Boundaries in 2D

Consider the set of points $\boldsymbol{r} = (x(u), y(u))$ where u is a continuous parameter, taking all values in the region R in the u-line and x and y are continuous functions of u in R. For every value of u in R there is a corresponding point (x,y) on a boundary S in 2D space. Hence we have a mapping from R to the boundary S.

<u>Example 1</u>

The boundary defined by

$$\boldsymbol{r} = (2\cos(\theta), \sin(\theta))$$

where -π≤θ≤π and is the boundary of an ellipse with major radius 2 and minor radius 1, centred at the origin.

Open and Closed Boundaries

A boundary S is said to be *open* if every pair of points that are not on S can be joined by a continuous line which does not cross S.

A boundary S is said to be *closed* if it divides space into two distinct regions; an interior region D and an exterior region E. That is that any continuous line joining any point in D with any point in E must cross the boundary S at least once.

<u>Example 1</u>

The surface of an ellipse, defined in example 1 above, is clearly a closed boundary.

<u>Example 2</u>

The boundary defined by

$$\boldsymbol{r} = (2\cos(\theta), \sin(\theta))$$

where 0≤θ≤ is half of an ellipse and it is an open surface.

Unit normal vector

For the boundary defined above, $\boldsymbol{r} = (x(u), y(u))$, at any point on the boundary, we can define the directional derivative

$$\frac{\partial \boldsymbol{r}}{\partial u} = \boldsymbol{r_u} \ .$$

The vector $[-\boldsymbol{r}_u[2], \boldsymbol{r}_u[1]] / |\boldsymbol{r_u}|$ is a normal to the boundary at $\boldsymbol{r}$.

The normal to a boundary may point in either direction; if $\boldsymbol{n}$ is a normal to a surface then so is $-\boldsymbol{n}$. Normally the sides of the boundary are labelled so that there is no ambiguity. In the case of a closed boundary, then the convention is that the normal points outward from the boundary; the outer surface is labelled as the positive boundary. A boundary is said to be *oriented* once the positive side has been determined.

Example 1

The boundary of an ellipse defined in example 1 is a closed ellipse.

$$\boldsymbol{r} = (2\cos(\theta), \sin(\theta))$$

Hence $\boldsymbol{r}_\theta = (-2\sin(\theta), \cos(\theta))$

$|\boldsymbol{r}_\theta| = \sqrt{4\cos^2(\theta) + \sin^2(\theta)} = \sqrt{3\cos^2(\theta) + 1}$.

Hence the normal to the boundary is $\frac{(2\sin(\theta), \cos(\theta))}{\sqrt{3\cos^2(\theta)+1}}$

Example 2

The surface is the upper half of the boundary already defined in example 1. The normal is defined similarly. However, since this is an open boundary, we can define either side as the positive boundary. We may reverse the normals, if we wish, as long as the orientation of the normals is applied consistently across the boundary.

If the unit normal is continuous at all points on the boundary then the boundary is said to be *smooth*. For example the surface of an ellipse is a smooth surface. A *simple boundary, piecewise smooth* or *regular* boundary is one which is composed of a set of smooth boundaries. For example the boundary of a square is a simple boundary.

Boundary length

Returning to our general definition of a boundary S:

$$\boldsymbol{r} = (x(u), y(u))$$

with u taking a range of values so that the whole of S is covered. The boundary length is given by the formula

$$\text{boundary length of } S = \int \boldsymbol{r}_u \, du$$

Example 1

The boundary length of the ellipse of example 1 is given by the formula

$$\int_{-\pi}^{\pi} |\boldsymbol{r}_\theta| d\theta = \int_{-\pi}^{\pi} \sqrt{3\cos^2(\theta) + 1}\, d\theta\,.$$

Such an integral is called an elliptic integral and it can only be solved numerically or by a formula.