Vector Arithmetic

In applied mathematics and physics and engineering, vectors often have two components to represent for example planar motion or – more likely – have three components to represent for example the real three-dimensional world. In this document we consider the representation of a vector and the most important operations on vectors – addition/subtraction and multiplication – and the size of a vector.

Representation of a vector[1]

The physical meaning of a vector is that it is a quantity that has both magnitude and direction. This is often diagrammatically-represented by an arrow, its angle representing the direction and its length representing its magnitude. A vector in a two-dimensional system a vector can be resolved into two perpendicular components; one in the x-direction and one in the y-direction. A vector in a three-dimensional system a vector can be resolved into three perpendicular components; one in the x, y and z-directions.

Vector Addition and Subtraction

In general mathematics, a vector can have any size. Vector addition and subtraction for the physical vectors considered in this document follow the same rules as those for general vectors in mathematics[2]; it simply involves the component-wise addition or subtraction.

For example

$$\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

and

$$\begin{pmatrix}7\\1\\3\end{pmatrix}-\begin{pmatrix}2\\-2\\5\end{pmatrix}=\begin{pmatrix}5\\3\\-2\end{pmatrix}.$$

Scalar or Dot Product

The scalar or dot product of two vectors is the sum of the component-wise products.

For example

$$\begin{pmatrix}1\\5\end{pmatrix}.\begin{pmatrix}2\\-3\end{pmatrix}=1\times 2+5\times(-3)=-13$$

and

$$\begin{pmatrix}7\\1\\3\end{pmatrix}.\begin{pmatrix}2\\-2\\5\end{pmatrix}=7\times 2+1\times(-2)+3\times 5=27.$$

[1] Vectors and Scalars
[2] Matrix Arithmetic

The dot product of two vectors in the same direction is equal to the product of their magnitudes. The dot product of two perpendicular vectors is zero.

For example

$$\begin{pmatrix}1\\5\end{pmatrix}.\begin{pmatrix}2\\10\end{pmatrix} = 1\times 2 + 5\times 10 = 52$$

and

$$\begin{pmatrix}1\\1\\0\end{pmatrix}.\begin{pmatrix}2\\-2\\0\end{pmatrix} = 1\times 2 + 1\times(-2) + 0\times 0 = 0.$$

Cross Product

The cross product of two vectors results in a vector that is perpendicular to the plane of the two original vectors. The cross product therefore only makes sense in three dimensions (in practical setting). The definition of the cross product requires the understanding of the meaning of the determinant of a matrix[3].

For example

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}2\\-2\\0\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 2 & -2 & 0\end{vmatrix}$$

$$= \mathbf{i}\,(1\times 0-(-2)\times 0) - \mathbf{j}(1\times 0 - 2\times 0) + \mathbf{k}(1\times(-2) - 2\times 1)$$

$$= -4\,\mathbf{k} = (0,0,-4)$$

[3] Inverse of a 3x3 Matrix